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Schrödinger equation for joint bidirectional motion in time

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Abstract

The conventional time-dependent Schrödinger equation describes only unidirectional time evolution of the state of a physical system, i.e. forward or, less commonly, backward. This paper proposes a generalized quantum dynamics for the description of joint, and interactive, forward and backward time evolution within a physical system. The principal mathematical assumption for bidirectional evolution in general is that the space of states should be taken to be not merely a Hilbert space, but a more restricted entity known as a Kreĭn space, which is a complex Hilbert space with a Hermitian operator that has eigenvalues +1 and −1 only, and that therefore gives rise to an indefinite metric. The vector subspaces of states with positive or negative norm with respect to the indefinite metric will—for open channels—be construed to be states in forward or, respectively, backward evolution along the time axis. The quantum dynamics is generated by a pseudo-Hermitian Hamiltonian operator and conserves inner products with respect to the indefinite metric. Input and output states are defined in physically plausible ways such that the output comprises both reflected and transmitted states from a zone of interaction in time; a unitary transformation between input and output states is obtained from the pseudounitary transformation between the initial and final states. Three applications are studied: (1) a formal theory of collisions in terms of perturbation theory; (2) a relativistically invariant quantum field theory for a system that kinematically comprises the direct sum of two quantized real scalar fields, such that one subfield evolves forward and the other backward in time, and such that there is dynamical coupling between the subfields; (3) an argument that in the latter field theory, the dynamics predicts that in a range of values of the coupling constants, the expectation value of the vacuum energy of the universe is forced to be zero to high accuracy.

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1. Introduction

The usual time-dependent Schrödinger equation is

$$\left[\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial t} + H\right]\Phi(t) = 0 \tag{1}$$

where $\Phi(t)$ is a time-evolving state vector in a Hilbert space, and H is a Hermitian Hamiltonian operator. This equation has the property that it describes just the unidirectional evolution of the state of a physical system from one time to another time that can be later or earlier than the first. Our principal objective herein is to construct a more general form of quantum mechanics that can describe a physical system in which part of the system evolves forward in time, while the remaining part evolves backward in time, and such that the two parts can interact.

The argument proceeds from the observation that such a formalism can be inferred from the quantum mechanics of two known physical systems: the first is the description, by a time-independent Schrödinger equation, of the evolution of a system along a space-like reaction coordinate, and the second is the complex Klein–Gordon equation for the motion of a spinless particle in the presence of a fixed, transient vector potential field. We shall not present the theory associated with these cases in detail, but sketch the ideas in the following two paragraphs.

For the evolution of a steady-state physical system along a space-like reaction coordinate, we cite as an example the evolution of the reversible, collinear chemical reaction $A + BC \leftrightarrow AB + C$ in the centre-of-mass system (see [1–4]) or, more simply, reflection and transmission of a beam of structureless particles from a potential barrier in one dimension. The second-order Schrödinger equation can be recast [5] as a coupled system of ordinary first-order equations, where the wavefunction is expanded in a set of vibrational states of the transverse coordinate. An indefinite metric matrix is derived from Wronskians, such that waves travelling forward along the reaction coordinate have positive norms, and waves travelling backward along the reaction coordinate have negative norms. The dynamics is governed by a Hamiltonian that is pseudo-Hermitian with respect to the metric, and hence conserves inner products with respect to the metric. The input comprises travelling waves (i.e. open channels) converging on the reaction zone, and the output comprises waves diverging from the reaction zone. A unitary S-matrix transforming input into output can be assembled from reflection and transmission matrices pertaining to open channels.

In the case of the Klein–Gordon equation, a Schrödinger-equation-like formalism has been derived by Feshbach and Villars [6], equation (2.15), and the following. The Hamiltonian proves to be pseudo-Hermitian with respect to an indefinite metric. The input comprises positive energy (and positive norm) states at large negative times, and negative energy (and negative norm) states at large positive times; the output comprises negative norm states at large negative times, and positive norm states at large positive times. It is straightforward, using the formalism developed by Bjorken and Drell [7], equations (9.6) and (9.20), to verify that a unitary S-matrix mapping input into output can be constructed from reflection and transmission coefficients.

A considerable selection of books has been published that is concerned with the physics and metaphysics of time, irreversibility, time's arrow and so on. A representative list comprises [8–17]. The book by Zeh [12] has a long list of references on the subject of its title, and much quantitative discussion; Zeh has put a preliminary version of the fourth edition of his book online[1].

[1] www.time-direction.de.

A line of investigation related to the present one was initiated by Schrödinger [18], as elaborated by Aebi [19, 20], and other works referenced therein. The generic idea of this line is to consider the evolution of diffusion or quantum processes, for which partial information on the state of the system is given at each of two finite times, and to infer the likeliest state of the system at intermediate times. Aharonov *et al* [21] and Reznik *et al* [22] constructed a time-symmetric quantum mechanics that utilizes information about the state of a system at both ends of a time interval to infer the expected results of measurements at an intermediate time. These investigations did not attempt to generalize quantum mechanics as is done here, but recast the existing physical laws in an alternate form. Perhaps the closest predecessor theory to that presented herein is the discussion/analysis of the problem of two-point boundary conditions in quantum mechanics by Schulman ([13], ch 5.3). Schulman's work is discussed in [12] (ch 5.3). In particular, Schulman ([13], p 184) introduces 'subspace boundary value problems' as a category of two-time boundary conditions; nevertheless, Schulman's quantum dynamics uses a Hermitian Hamiltonian, and correspondingly does not introduce an indefinite metric, so that his proposed theory does not conserve probability in the sense that will be done here. Schulman [23], plus a directed comment by Casati *et al* [24] and reply by Schulman [25], dealt with a classical mechanics construction of opposite thermodynamic arrows of time.

Cramer [26] has developed a 'transactional' interpretation of quantum mechanics that involves the presence of advanced as well as retarded interactions that are invoked to relieve some of the counterintuitive nonlocality involved in the collapse of the wavefunction. Cramer, however, does not introduce a generalized dynamics associated with the transactional interpretation, and makes predictions that do not differ from those of standard quantum mechanics ([26], ch III.B, last paragraph). I infer also that Cramer presumes that the strength of the interactions of the advanced waves with ordinary matter is the same as, or roughly comparable to, that of retarded waves. In the theory described below, interactions between the forward- and backward-evolving subspaces are presumed on physical grounds to be very small compared to, say, electromagnetic interactions within each subspace.

A subject that is employed in the mathematics used herein is the study of infinite-dimensional complex vector spaces that are endowed with a nondegenerate, sesquilinear inner product that gives rise to an indefinite metric. In quantum field theory, this subject was first studied by Dirac [27], and in mathematics by Pontrjagin [28]. The former area was a subject of interest in the 1940s to the early 1960s, as reviewed in [29]; the latter subject is still an area of mathematical interest, see [30].

The remainder of this paper is organized as follows. In section 2 we formulate a quantum dynamics, in the form of a Schrödinger equation and some rules for interpreting the associated mathematics, that can treat physical systems in which joint, and interactive, motion or evolution in both directions in time can occur. Section 3 derives a formal theory of scattering, i.e. transition operators and S-matrices, for collision processes with a time-independent Hamiltonian governing the dynamics. Section 4 presents the basics of a physical system comprising the direct sum of two interacting quantized real scalar fields; the theory is shown to be relativistically invariant, and perturbation theory is applied to a case of two-body collisions. Section 5 concludes the paper with a discussion of some of the ideas presented herein, and with a quantitative argument to the effect that in a suitable range of parameter values of the field theory of section 4, the expectation value of the vacuum energy of the universe necessarily vanishes to high accuracy. The appendix shows how to obtain transition rates from transition operators.

We emphasize that the statement given herein of a Schrödinger equation to describe bidirectional motion in time is incomplete: important, but derivative, theoretical aspects, such as a manifestly covariant perturbation scheme for the quantum field theory of section 4, and modifications of quantum measurement theory, including an analysis of wavefunction 'collapse', etc remain to be worked out.

2. Quantum mechanics of bidirectional motion

In this section, we shall propose a formalism that accomplishes the paper's title objective. The principal mathematical idea is to introduce a state space with a nondegenerate inner product that yields an indefinite metric, and correspondingly, a pseudo-Hermitian Hamiltonian to govern the dynamics. The attendant physical interpretation will posit that state space comprises the direct sum of two orthogonal subspaces, such that one has a positive definite norm and the other a negative definite norm; for open channels, these two subspaces will correspond to those states of motion of the system that evolve forward and backward in time, respectively.

Some of the mathematical community presently designates a state space of the above type, with a suitable topology, as a Kreĭn space (described, with references, in the encyclopedia [31], vol 5, p 303), named for the Ukrainian mathematician M G Kreĭn: see [32] for a description of Kreĭn's work in this area, and [30] for the theory of Kreĭn spaces. An earlier designation, Nevanlinna space (mentioned in [29], section 1), now applies to a different entity [33][2]. The properties of matrices in finite-dimensional vector spaces with an indefinite metric are discussed in [34]. An alternate formulation of the latter class of spaces has been called 'complex symplectic geometry' (see [35]), although this usage conflicts with an earlier development ([36], p 23, definition 1), in that the extension of symplectic geometry from the real coefficient field to the complex field entails a sesquilinear and, implicitly, a bilinear form in the respective definitions. The mathematical physics community for the most part seems to have used the designation 'space with an indefinite metric', although the name 'Kreĭn space' sometimes appears [37]; the designation 'pseudo-Hilbert space' [38, 39] was used rarely.

Beginning with the work of Dirac [27] and Pauli [40], a substantial body of work on quantum field theory was done that dealt with state spaces with an indefinite metric, as reviewed in [29]. There is little overlap between this theoretical work and that presented below: (1) we shall not introduce anomalous commutators for the creation and destruction operators associated with a quantum field; (2) we shall (in section 4) deal with a field theory for which a complete quantum state is a vector in a space that is made up of the direct sum of the Fock spaces of two conventional field theories; (3) the S-matrix will be obtained, not by the mapping of the system's state at $t = -\infty$ into the state at $t = +\infty$ as input into output, but as a mapping with a different choice of input and output such that probability is conserved and the S-matrix is unitary.

More recent work on associated mathematical physics, such as [37], will not be needed herein as the nonlocal input/output conditions in time suggest a different approach.

We begin with a Hilbert space $\mathcal{H}$, with vectors denoted as $\psi \in \mathcal{H}$, and a sesquilinear product $\langle .,. \rangle$ with the standard inner product (unit metric matrix) form, such that

$$\langle \psi_1, \psi_2 \rangle = (\psi_1)^\dagger \psi_2 \tag{2a}$$

$$\langle \psi_1, \alpha\psi_2 + \beta\psi_3 \rangle = \alpha\langle \psi_1, \psi_2 \rangle + \beta\langle \psi_1, \psi_3 \rangle \tag{2b}$$

$$\langle \psi_1, \psi_2 \rangle = \langle \psi_2, \psi_1 \rangle^*. \tag{2c}$$

[2] See p 8. The *Nevanlinna class*—which I presume is synonymous with *Nevanlinna space*—is called N and comprises those meromorphic functions on the open unit disc in the complex plane that have a bounded *characteristic function* (in the work cited).

We postulate further that $\mathcal{H}$ is equivalent to the direct sum of exactly two subspaces $\mathcal{H}^F$ and $\mathcal{H}^B$ with corresponding Hermitian projection operators P^F and P^B, such that

$$(P^Y)^\dagger = P^Y \tag{3a}$$

$$P^F + P^B = I \tag{3b}$$

$$P^Y P^{Y'} = P^Y \delta^{YY'} \tag{3c}$$

$$P^F \mathcal{H} = \mathcal{H}^F \oplus 0^B \tag{3d}$$

$$P^B \mathcal{H} = 0^F \oplus \mathcal{H}^B \tag{3e}$$

where I is the identity operator in $\mathcal{H}$, Y and Y' can each be F or B, and 0^Y is the zero subspace in $\mathcal{H}^Y$. We shall use I^F and I^B as the identity operators in the respective subspaces. We shall not distinguish between $\mathcal{H}$ and the direct sum $\mathcal{H}^F \oplus \mathcal{H}^B$; accordingly, if we define for $Y = F, B$

$$\psi^Y = (P^Y \psi)|_{\mathcal{H}^Y} \in \mathcal{H}^Y \tag{4}$$

we can describe ψ in block column matrix form as

$$\psi = \begin{bmatrix} \psi^F \\ \psi^B \end{bmatrix}. \tag{5}$$

We now define an operator η that engenders an indefinite metric,

$$\eta = P^F - P^B \tag{6}$$

and an associated inner product $(.; .)$ as

$$(\psi_1; \psi_2) = \langle \psi_1, \eta\psi_2 \rangle. \tag{7}$$

The η-adjoint $T^\ddagger$ of an operator T acting on $\mathcal{H}$ is defined as the unique operator that satisfies

$$\left(T^\ddagger \psi_1; \psi_2\right) = (\psi_1; T\psi_2) \tag{8}$$

for all $\psi_1, \psi_2 \in \mathcal{H}$. An operator T will be called pseudounitary if it preserves η-products, that is for all $\psi_1, \psi_2 \in \mathcal{H}$ we have

$$(T\psi_1; T\psi_2) = (\psi_1; \psi_2) \tag{9}$$

and pseudo-Hermitian if $T^\ddagger = T$, that is

$$(T\psi_1; \psi_2) = (\psi_1; T\psi_2). \tag{10}$$

If we revert to the block matrix form of equation (5) we infer that

$$(\psi_1; \psi_2) = \psi_1^\dagger \eta \psi_2 = \left(\psi_1^F\right)^\dagger \psi_2^F - \left(\psi_1^B\right)^\dagger \psi_2^B. \tag{11}$$

Also, if for an operator T we define

$$T^{YY'} = \left(P^Y T P^{Y'}\right)\Big|_{\mathrm{Hom}[\mathcal{H}^Y \leftarrow \mathcal{H}^{Y'}]} \tag{12}$$

where $\mathrm{Hom}[\mathcal{H}^Y \leftarrow \mathcal{H}^{Y'}]$ is the set of complex-linear mappings (i.e. homomorphisms) from $\mathcal{H}^{Y'}$ into $\mathcal{H}^Y$, then we have, in block matrix notation,

$$T = \begin{bmatrix} T^{FF} & T^{FB} \\ T^{BF} & T^{BB} \end{bmatrix}. \tag{13}$$

If T is pseudo-Hermitian we have

$$T = T^\ddagger = \eta T^\dagger \eta = \begin{bmatrix} (T^{FF})^\dagger & -(T^{BF})^\dagger \\ -(T^{FB})^\dagger & (T^{BB})^\dagger \end{bmatrix}. \tag{14}$$

If T is pseudounitary we have the (we presume both left and right) inverse T^{-1} that satisfies

$$T^{-1} = T^{\ddagger} \tag{15}$$

and, therefore,

$$(T^{FF})^{\dagger}(T^{FF}) - (T^{BF})^{\dagger}(T^{BF}) = I^{F} \tag{16a}$$

$$-(T^{FB})^{\dagger}(T^{FF}) + (T^{BB})^{\dagger}(T^{BF}) = 0 \tag{16b}$$

$$-(T^{FB})^{\dagger}(T^{FB}) + (T^{BB})^{\dagger}(T^{BB}) = I^{B}. \tag{16c}$$

Let it be given that T is pseudounitary and that T^{BB} has an inverse $(T^{BB})^{\iota}$ within $\mathcal{H}^{B}$, in that

$$(T^{BB})^{\iota}T^{BB} = I^{B} = T^{BB}(T^{BB})^{\iota}. \tag{17}$$

Then the block operator-matrix $\tilde{U}(T)$, defined as

$$\tilde{U}(T) = \begin{bmatrix} T^{FF} - T^{FB}(T^{BB})^{\iota}T^{BF} & T^{FB}(T^{BB})^{\iota} \\ -(T^{BB})^{\iota}T^{BF} & (T^{BB})^{\iota} \end{bmatrix} \tag{18}$$

can, with the aid of equation (16), be proved to be unitary on the left, and similarly for right unitarity. A more complicated procedure is needed to extract a unitary S-matrix when asymptotic closed channels are present—see section 3.

A time-dependent vector $\psi(t) \in \mathcal{H}$, that is an eigenvector of η with eigenvalue +1 (respectively -1) will—closed channels excepted—be considered to evolve forward (respectively backward) in time. The expectation value $(\psi(t); \psi(t))$ of a general state $\psi(t)$ will be construed as the integrated probability current crossing the complete space-like surface time $= t$. The operator η is therefore a kind of velocity operator, describable as the derivative of dynamical causation time with respect to kinematical time, and can take only the values +1 and -1. This interpretation therefore addresses the question of the velocity of objective flow of time posed in [14] (p 13).

We proceed from kinematics to a theory of quantum dynamics. Let P^{F} and P^{B} be time independent, $H(t)$ be a Hamiltonian that is pseudo-Hermitian at each instant, and $\Phi(t) \in \mathcal{H}$ a kinematically allowable family of state vectors, described parametrically by dependence on time. The time evolution of a dynamically allowable family of quantum states $\Phi(t)$ is governed by the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = H(t)\Phi(t). \tag{19}$$

When both $\Phi_1(t)$ and $\Phi_2(t)$ are solutions of equation (19), their η-product as in equation (7) will be independent of time. Furthermore, if another operator Z is independent of time and commutes with $H(t)$ for all times, then $(\Phi_1(t); Z\Phi_2(t))$ also is a constant in time.

Suppose now that we have obtained a complete set of solutions of equation (19) across any desired time interval $t_- \leqslant t \leqslant t_+$; equivalently, we have for each closed interval $[t_-, t_+]$ in time a linear operator $\Upsilon(t_+, t_-)$ such that

$$\Phi(t_+) = \Upsilon(t_+, t_-)\Phi(t_-) \tag{20}$$

for any initial $\Phi(t_-) \in \mathcal{H}$. One can show that $\Upsilon(t_+, t_-)$ is pseudounitary. We define the input to the physical process taking place to be the blocked vector

$$\Phi_{\mathrm{in}}(t_-, t_+) = \begin{bmatrix} (P^{F}\Phi(t_-))|_{\mathcal{H}^{F}} \\ (P^{B}\Phi(t_+))|_{\mathcal{H}^{B}} \end{bmatrix} \tag{21}$$

and the output to be

$$\Phi_{\mathrm{out}}(t_+, t_-) = \begin{bmatrix} (P^{F}\Phi(t_+))|_{\mathcal{H}^{F}} \\ (P^{B}\Phi(t_-))|_{\mathcal{H}^{B}} \end{bmatrix}. \tag{22}$$

One can now show that, following equation (18), the operator $\tilde{U}(\Upsilon(t_+, t_-))$ is unitary and that

$$\Phi_{\text{out}}(t_+, t_-) = \tilde{U}(\Upsilon(t_+, t_-))\Phi_{\text{in}}(t_-, t_+). \tag{23}$$

In analysing any physical process taking place in the interval $[t_-, t_+]$ we assume that the input state is given, known or controllable. We can obviously multiply $\Phi(t)$ for all t by a constant factor, with the desired outcome

$$\langle \Phi_{\text{in}}(t_-, t_+), \Phi_{\text{in}}(t_-, t_+) \rangle = +1 \tag{24}$$

(note the use of the Hilbert space norm), so that equation (23) implies

$$\langle \Phi_{\text{out}}(t_+, t_-), \Phi_{\text{out}}(t_+, t_-) \rangle = +1. \tag{25}$$

We therefore have in our possession the bare bones of a probability interpretation for the proposed scheme of kinematics and dynamics.

We remark that the above interpretation as to what constitutes the input and what the output to a dynamical process requires modification if one or both ends of the time interval diverge: since the Hamiltonian can have nonreal eigenvalues, care must be taken to avoid the divergent solutions associated with these closed-channel states. A class of such problems is dealt with in the S-matrix formalism of the following section.

We continue to use the assumptions of the previous paragraph, including the normalization condition (24) on the input. Let $Z(t)$ be a pseudo-Hermitian operator, and define the expectation value $[Z(t)]_{\text{Av}}$ of $Z(t)$ for each t, with the system in the state $\Phi(t)$, in the standard manner:

$$[Z(t)]_{\text{Av}} = (\Phi(t); Z(t)\Phi(t)). \tag{26}$$

If $Z(t) = I$, the expectation value is just the conserved η-norm of $\Phi(t)$, which can be anywhere between +1 and -1. As mentioned above, we shall refer to this quantity as the net probability current at time $= t$ in spacetime. This current is more closely analogous to an electric charge than to a spatial electric current: the electric charge is the integrated value of the zeroth (time) component of the 4-vector electric current density over a surface $t =$ constant; considered in this way, a total electric charge amounts to a net electric current crossing a complete space-like surface. (Considered in another way, the state vector stays put at any given time; it is we who are moving through time, and hence we see a changing state vector and thereby net currents of physical quantities as probability, electric charge, etc.) In the present case, we do not define a 4-vector probability current density, but simply take as a physical axiom that what is normally called 'probability' is now to be regarded as the net probability current associated with a quantum state at a given time. For a general pseudo-Hermitian $Z(t)$ its expectation value with respect to $\Phi(t)$ will be real, and will be taken to have the physical meaning of the net current, or flow, or transport, of the physical quantity associated with $Z(t)$ across the chosen complete space-like surface, as that surface moves forward in time with velocity +1. Note that since the metric or velocity operator η can have only the dimensionless eigenvalues +1 and -1, a (likely unphysical) density that gives rise to the current associated with a $Z(t)$, which might take the form of $\langle \Phi(t), Z(t)\Phi(t) \rangle$, and the expectation value itself, have the same physical dimensions.

3. Formal scattering theory

In this section, we shall develop a theory of scattering patterned after the developments in [41], ch 2.5, and [42], sections 16.2 and 16.3. This formalism treats the time and energy coordinates in a different way than it treats space and momentum, such that it applies whether or not the underlying dynamics is relativistically invariant; a corresponding disadvantage is the resulting lack of manifest relativistic invariance of the terms in the perturbation theory expansion for the S-matrix in cases such as the field theory of section 4, which is shown there to be relativistically invariant in its Hamiltonian form. The formalism will generalize the conventional one in two respects: first, evolution in both directions in time will be included, and second, the zeroth-order Hamiltonian will be permitted to have some nonreal eigenvalues in its spectrum—these correspond to asymptotically closed channels.

Suppose that the Hamiltonian is time independent and has the form

$$H = H^{[0]} + H^{[1]} \tag{27}$$

where for both $\sigma = 0, 1$

$$H^{[\sigma]} = \begin{bmatrix} H^{[\sigma]FF} & -H^{[\sigma]BF\dagger} \\ H^{[\sigma]BF} & H^{[\sigma]BB} \end{bmatrix} \tag{28}$$

where the diagonal-block operators are Hermitian, and $H^{[\sigma]BF}$ is unrestricted within the bounds of physical reasonableness. We shall adopt the picture that for large negative times and large positive times the effects of $H^{[1]}$ are negligible: the very early, as well as the very late, quantum state can be envisioned as comprising superpositions of states, each of which describes two spatially widely separated wave packets, such that each packet represents an entity that does not interact with its partner, and the overall state is a solution of the Schrödinger equation with $H^{[0]}$ as the Hamiltonian.

We shall argue first that for nontransient transitions to take place, the real sector of the eigenvalue spectrum of the unperturbed Hamiltonian must, in effect, be positive for both the states of forward motion in time (FMT) and the states of backward motion in time (BMT). In the Hamiltonian given above, both $H^{[0]}$ and $H^{[1]}$ are to be time independent; hence energy is conserved—there will be a delta-function in overall energy that arises in the results below. A time-independent Hamiltonian that gives rise to nontransient transitions between FMT and BMT states therefore requires that these two sets of states have the real sectors of their respective eigenvalue spectra overlap. We therefore abandon the picture that states in BMT correspond to negative energy states: in general, both FMT and BMT states with real energy eigenvalues will be assumed to have positive energies, or at least energies that are bounded below but not above along the real axis.

Next, we specialize $H^{[0]}$ and η so that they have the properties that, although the state space may be infinite dimensional, permit them to be jointly reduced to that canonical form for finite-dimensional matrices—described in [34], p 33, theorem 3.3—which occurs when the minimal polynomial for the Hamiltonian matrix is a product of distinct linear factors. In particular, we assume that the eigenstates of $H^{[0]}$ form a complete, orthogonal set with respect to the underlying Hilbert space. Explicitly, suppose that there is a direct-sum decomposition of the full state space $\mathcal{H}$ such that

$$\mathcal{H} = \mathcal{H}^R \oplus \mathcal{H}^N \tag{29a}$$

$$\mathcal{H}^R = \mathcal{H}^{R,F} \oplus \mathcal{H}^{R,B} \tag{29b}$$

$$\mathcal{H}^N = \mathcal{H}^{N,1} \oplus \mathcal{H}^{N,2} \tag{29c}$$

and a basis compatible with this decomposition, such that $H^{[0]}$ is diagonal, η is diagonal in the subspace $\mathcal{H}^R$ belonging to the real eigenvalue spectrum of $H^{[0]}$, and η has a simple, block off-diagonal form in the subspace $\mathcal{H}^N$ belonging to the nonreal eigenvalue spectrum of $H^{[0]}$. Furthermore, all vectors in $H^{R,F}$ (respectively $H^{R,B}$) are eigenstates of η with eigenvalue +1 (respectively -1). Since each nonreal eigenvalue must have a complex conjugate partner, we can take $H^{N,1}$ and $H^{N,2}$ to be copies of one another; the former being associated with nonreal eigenvalues with negative imaginary part, the latter with nonreal eigenvalues with positive imaginary part. In matrix form, therefore, we have assumed the existence of an invertible linear transformation T taking $H^{[0]}$ expressed in an arbitrary basis into a basis so that we have

$$T^{-1}H^{[0]}T = \mathrm{diag}(\Delta^{R,F}, \Delta^{R,B}, \Delta^{N,1}, \Delta^{N,2}) \tag{30a}$$

$$T^{\dagger}\eta T = \begin{bmatrix} I^{R,F} & 0 & 0 & 0 \\ 0 & -I^{R,B} & 0 & 0 \\ 0 & 0 & 0 & U^{N,12} \\ 0 & 0 & U^{N,21} & 0 \end{bmatrix}. \tag{30b}$$

In the above, $\Delta^{R,F}$ and $I^{R,F}$ are a real diagonal and the unit matrix, respectively, acting on $\mathcal{H}^{R,F}$; $\Delta^{R,B}$ and $I^{R,B}$ are a real diagonal and the unit matrix, respectively, acting on $\mathcal{H}^{R,B}$; $\Delta^{N,1}$ is a diagonal matrix with diagonal elements having negative imaginary parts, acting on $\mathcal{H}^{N,1}$; $U^{N,21}$ is the unitary mapping of $\mathcal{H}^{N,1}$ onto $\mathcal{H}^{N,2}$ that takes an eigenstate with eigenvalue Λ (having, we have assumed, $\mathrm{Im}(\Lambda) < 0$) with respect to $\Delta^{N,1}$ into a partner eigenstate having an eigenvalue Λ^* with respect to $\Delta^{N,2}$; and $U^{N,12}$ is the inverse of $U^{N,21}$, in that

$$U^{N,21} = (U^{N,12})^{\dagger} \tag{31a}$$

$$U^{N,21}U^{N,12} = I^{N,2} \tag{31b}$$

$$U^{N,12}U^{N,21} = I^{N,1} \tag{31c}$$

$$U^{N,21}\Delta^{N,1}U^{N,12} = (\Delta^{N,2})^{\dagger} \tag{31d}$$

where $I^{N,1}$ and $I^{N,2}$ are the unit matrices in the spaces $\mathcal{H}^{N,1}$ and $\mathcal{H}^{N,2}$, respectively. Equations (31) are in accord with the pseudo-Hermitian property for $H^{[0]}$.

We remark that the above special form of $H^{[0]}$ excludes all so-called 'ghost' states—see [29], p 14, for definitions, and [34], p 33, theorem 3.3 for the joint canonical form of a general pseudo-Hermitian matrix and the metric matrix in the finite-dimensional case.

We want to find the Green function for $H^{[0]}$ such that both open-channel (R, F) states and closed-channel $(N, 1)$ states evolve forward in time, while open-channel (R, B) and closed-channel $(N, 2)$ states evolve backward in time. If we put $\hbar = 1$, $G^{[0]}(t - t')$ should satisfy

$$\left[\mathrm{i}\frac{\partial}{\partial t} - H^{[0]}\right] G^{[0]}(t - t') = \delta(t - t'). \tag{32}$$

The desired solution is

$$\begin{aligned} G^{[0]}(t - t') = \mathrm{diag}(&-\mathrm{i}\theta(t - t')\exp[-\mathrm{i}(t - t')\Delta^{R,F}],\ \mathrm{i}\theta(t' - t)\exp[-\mathrm{i}(t - t')\Delta^{R,B}], \\ &-\mathrm{i}\theta(t - t')\exp[-\mathrm{i}(t - t')\Delta^{N,1}],\ \mathrm{i}\theta(t' - t)\exp[-\mathrm{i}(t - t')\Delta^{N,2}]) \end{aligned} \tag{33}$$

where θ is the unit step function. The Fourier transform of $G^{[0]}$ is

$$\tilde{G}^{[0]}(E) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}sE)G^{[0]}(s)\,\mathrm{d}s \tag{34}$$

$$\begin{aligned} &= \mathrm{diag}([(E + \mathrm{i}\epsilon)I^{R,F} - \Delta^{R,F}]^{-1}, [(E - \mathrm{i}\epsilon)I^{R,B} - \Delta^{R,B}]^{-1}, \\ &\quad [EI^{N,1} - \Delta^{N,1}]^{-1}, [EI^{N,2} - \Delta^{N,2}]^{-1}). \end{aligned} \tag{34a}$$

Adding $+\mathrm{i}\epsilon$ (respectively, $-\mathrm{i}\epsilon$) to E effects the usual small displacement of the poles of the integrand down (respectively, up) from the real axis in the complex E-plane when recovering $G^{[0]}(t - t')$ from $\tilde{G}^{[0]}(E)$; no displacement is needed for poles off the real axis. If there is a nonzero gap between the entire nonreal spectrum and the real axis, a very small raising or lowering of the nonreal spectrum in the E-plane will not affect the result in this subspace of $\mathcal{H}$; in such a case, we can give an abbreviated formula for $\tilde{G}^{[0]}(E)$, that is

$$\tilde{G}^{[0]}(E) = (EI + \mathrm{i}\epsilon\eta - H^{[0]})^{-1} \tag{35}$$

where I is the unit operator in $\mathcal{H}$.

We shall now specify a complete, orthogonal (in the Hilbert space sense) set of eigenfunctions of $H^{[0]}$. Let $\mathcal{S}^{R,F}$ (respectively $\mathcal{S}^{R,B}$) denote the subset of real eigenvalues of $H^{[0]}$ such that the corresponding eigenstates are also eigenstates of η with eigenvalue $+1$ (respectively -1). Let $\mathcal{S}^{N,1}$ denote the set of those eigenvalues of $H^{[0]}$ having negative imaginary part, and $\mathcal{S}^{N,2}$ be the set of complex conjugate points of those in $\mathcal{S}^{N,1}$. We shall assume that $\mathcal{S}^{N,1}$ is, or can be approximated by, a discrete spectrum; conceivably, however, there may exist $H^{[0]}$ such that the corresponding set $\mathcal{S}^{N,1}$ has a nondiscrete topology, e.g. a subset of a curve in $\mathbb{C}$.

We denote a state in the basis leading to the matrix form of equation (30) as $\Psi^{[0]Z,Y}_{\Lambda\gamma}$. The index Z can take the values R or N, and for $Z = R$, Y can take the values F or B, while for $Z = N$, Y can take the values 1 or 2. Let α_Y be defined as

$$\alpha_Y = \begin{cases} +1 & \text{if} \quad Y = F \\ -1 & \text{if} \quad Y = B \end{cases} \tag{36}$$

Λ be the eigenvalue of $H^{[0]}$, and γ (an index which is implicitly dependent on the other quantum numbers) label degenerate states with respect to $H^{[0]}$. We note the following behaviour of these eigenstates under the action of η:

$$\eta\Psi^{[0]R,Y}_{\Lambda,\gamma} = \alpha_Y \Psi^{[0]R,Y}_{\Lambda,\gamma} \tag{37a}$$

$$\eta\Psi^{[0]N,1}_{\Lambda^*\gamma} = \Psi^{[0]N,2}_{\Lambda\gamma} \tag{37b}$$

$$\eta\Psi^{[0]N,2}_{\Lambda^*\gamma} = \Psi^{[0]N,1}_{\Lambda\gamma}. \tag{37c}$$

The Hilbert space orthonormality of the states and the completeness relation are as follows:

$$\left(\Psi^{[0]Z',Y'}_{\Lambda'\gamma'}\right)^\dagger \left(\Psi^{[0]Z,Y}_{\Lambda\gamma}\right) = \delta^{Z'Z}\delta^{Y'Y} \begin{cases} \delta(\Lambda' - \Lambda)\delta_{\gamma'\gamma} & \text{if} \quad Z = R \\ \delta_{\Lambda'\Lambda}\delta_{\gamma'\gamma} & \text{if} \quad Z = N \end{cases} \tag{38a}$$

$$I = \sum_{Y=F,B} \int_{E\in\mathcal{S}^{R,Y}} \sum_{\gamma} \Psi^{[0]R,Y}_{E\gamma} \left(\Psi^{[0]R,Y}_{E\gamma}\right)^\dagger \mathrm{d}E + \sum_{\Lambda\in\mathcal{S}^{N,1}} \sum_{\gamma} \left[\Psi^{[0]N,1}_{\Lambda\gamma} \left(\Psi^{[0]N,1}_{\Lambda\gamma}\right)^\dagger + \Psi^{[0]N,2}_{\Lambda^*\gamma} \left(\Psi^{[0]N,2}_{\Lambda^*\gamma}\right)^\dagger \right]. \tag{38b}$$

We find the time-dependent, open-channel solutions of the Schrödinger equation with $H^{[0]}$ as Hamiltonian, to be

$$\Phi^{[0]R,Y}_{E\gamma}(t) = \exp(-\mathrm{i}Et)\Psi^{[0]R,Y}_{E\gamma}. \tag{39}$$

Then the full scattering wavefunction $\Phi^{R,Y}_{E\gamma}(t)$ with input as $\Phi^{[0]R,Y}_{E\gamma}(t)$ for $t \to -\alpha_Y\infty$ satisfies the integral equation

$$\Phi^{R,Y}_{E\gamma}(t) = \Phi^{[0]R,Y}_{E\gamma}(t) + \int_{-\infty}^{+\infty} G^{[0]}(t - t_1) H^{[1]} \Phi^{R,Y}_{E\gamma}(t_1)\, \mathrm{d}t_1. \tag{40}$$

We presume that this equation can be solved by unlimited Neumann iterations, with the result

$$\Phi^{R,Y}_{E\gamma}(t) = \Phi^{[0]R,Y}_{E\gamma}(t) + \int_{-\infty}^{+\infty} G^{[0]}(t-t_1)H^{[1]}\Phi^{[0]R,Y}_{E\gamma}(t_1)\,\mathrm{d}t_1$$
$$+\sum_{j=2}^{\infty}\int\cdots\int_{-\infty}^{+\infty}\mathrm{d}t_1\cdots\mathrm{d}t_j\,G^{[0]}(t-t_1)H^{[1]}$$
$$\times\left[\prod_{k=2}^{j}G^{[0]}(t_{k-1}-t_k)H^{[1]}\right]\Phi^{[0]R,Y}_{E\gamma}(t_j). \tag{41}$$

In the rhs of equation (41) let us now (i) use equation (39) for the zero-order wavefunctions, (ii) substitute the inverse of equation (34) for each entry $G^{[0]}(t_{k-1}-t_k)$ in the product in equation (41), (iii) change variables of integration from t_k to s_k (for $k = 2, \ldots, j$, while t_1 is unchanged) in the jth summand, where

$$s_k = t_{k-1} - t_k \qquad \text{for} \quad k = 2, \ldots, j \tag{42a}$$

so that

$$-t_j = -t_1 + \sum_{k=2}^{j} s_k, \tag{42b}$$

(iv) carry out the integrals over $s_2, \ldots, s_j$ in the jth summand, and (v) do the resulting integrals involving delta-functions in energy. We define the transition operator $T(E)$ as

$$T(E) = H^{[1]}\sum_{j=0}^{\infty}\left[\tilde{G}^{[0]}(E)H^{[1]}\right]^j \tag{43a}$$

$$= H^{[1]}\left[I - \tilde{G}^{[0]}(E)H^{[1]}\right]^{-1} \tag{43b}$$

$$= \left[I - H^{[1]}\tilde{G}^{[0]}(E)\right]^{-1}H^{[1]} \tag{43c}$$

where the zero power of an operator is the unit operator. Then equation (41) reduces to

$$\Phi^{R,Y}_{E\gamma}(t) = \exp(-\mathrm{i}Et)\Psi^{[0]R,Y}_{E\gamma} + \int_{-\infty}^{+\infty} G^{[0]}(t-t_1)\exp(-\mathrm{i}Et_1)T(E)\Psi^{[0]R,Y}_{E\gamma}\,\mathrm{d}t_1. \tag{44}$$

Let us now take the η-product of both sides of equation (44) with $\Phi^{[0]R,Y'}_{E'\gamma'}(t)$, while also inserting the unit operator, in the form of the rhs of equation (38b), following the Green function in the integrand of equation (44): using expression (33) for the Green function, we find, after some manipulation, that

$$\left(\Phi^{[0]R,Y'}_{E'\gamma'}(t);\,\Phi^{R,Y}_{E\gamma}(t)\right) = \alpha_{Y'}\delta^{Y'Y}\delta(E'-E)\delta_{\gamma'\gamma} - \mathrm{i}\left(\Psi^{[0]R,Y'}_{E'\gamma'};\,\eta T(E)\Psi^{[0]R,Y}_{E\gamma}\right)$$
$$\times\left[\delta^{Y'F}\int_{-\infty}^{t}\exp[\mathrm{i}(E'-E)t_1]\,\mathrm{d}t_1 + \delta^{Y'B}\int_{t}^{+\infty}\exp[\mathrm{i}(E'-E)t_1]\,\mathrm{d}t_1\right]. \tag{45}$$

The derivation from equation (45) of an expression for the transition probability per unit time is carried out in the appendix. If we define the inverse function to equation (36) as

$$\bar{Y}_\alpha = \begin{cases} F & \text{if} \quad \alpha = +1 \\ B & \text{if} \quad \alpha = -1 \end{cases} \tag{46}$$

then as $|t| \to \infty$, equation (45) has the limiting forms

$$\left(\Phi^{[0]R,Y'}_{E'\gamma'}(t);\,\Phi^{R,Y}_{E\gamma}(t)\right) \to \alpha_{Y'}\delta^{Y'Y}\delta(E'-E)\delta_{\gamma'\gamma} - 2\pi\mathrm{i}\delta(E'-E)$$
$$\times\,\delta^{Y'\bar{Y}_\alpha}\left(\Psi^{[0]R,Y'}_{E'\gamma'};\,\eta T(E)\Psi^{[0]R,Y}_{E\gamma}\right) \qquad \text{as} \quad t \to \alpha\infty. \tag{47}$$

We analyse equation (47) to determine the analogues of reflection and transmission coefficients, and assemble the results into an S-operator; that is we want to have

$$\left(\Phi^{[0]R,Y'}_{E'\gamma'}(t);\, \Phi^{R,Y}_{E\gamma}(t)\right) \to \begin{cases} \left(\Psi^{[0]R,Y'}_{E'\gamma'};\, S\Psi^{[0]R,Y}_{E\gamma}\right) & \text{for} \quad t \to +\alpha_{Y'}\infty \\ \left(\Psi^{[0]R,Y'}_{E'\gamma'};\, \Psi^{[0]R,Y}_{E\gamma}\right) & \text{for} \quad t \to -\alpha_{Y'}\infty. \end{cases} \tag{48}$$

On the basis of a comparison of equations (47) and (48), we proceed to define the S-operator as an entity that acts on, and only on, the subspace $\mathcal{H}^R$ of $\mathcal{H}$. We define I^R as the identity operator within $\mathcal{H}^R$, and X^{RR} as the restriction of a general operator $X : \mathcal{H} \to \mathcal{H}$ to the suboperator that maps to $\mathcal{H}^R \to \mathcal{H}^R$. We note that in the special cases treated here of an $H^{[0]}$ and η of the form of equation (30), $H^{[0]RR}$ and η^{RR} commute, and correspondingly $H^{[0]RR}$ is Hermitian. Then if we let

$$S = I^R - 2\pi\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}E''\big[\delta\big(E''I - H^{[0]}\big)\eta T(E'')\delta\big(E''I - H^{[0]}\big)\big]^{RR} \tag{49}$$

equations (47) and (48) are in accord.

It remains to prove that the S-operator acts unitarily within $\mathcal{H}^R$. In fact, we infer from equation (49) that

$$SS^\dagger - I^R = -2\pi\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}E\big[\delta\big(EI - H^{[0]}\big)\eta\big]^{RR}[\Xi(E)]^{RR}\big[\eta\delta\big(EI - H^{[0]}\big)\big]^{RR} \tag{50}$$

where, by definition,

$$\Xi(E) = T(E)\eta - \eta T(E)^\dagger + 2\pi\mathrm{i}T(E)\delta\big(EI - H^{[0]}\big)T(E)^\dagger. \tag{51}$$

Here we made use of the properties that the operator $\delta(EI - H^{[0]})$ has its cokernel and image contained within the subspace $\mathcal{H}^R$, and that, as a result of equations (30*a*) and (30*b*), $\delta(E - H^{[0]})$ is Hermitian as well as pseudo-Hermitian. We want to prove that $\Xi(E)$ equals the zero operator for all real values of E. To do this, we modify the argument that leads to [41], equation (5.29). The steps are very similar, except that now $\eta H^{[1]\dagger} = H^{[1]}\eta$, and we need the easily verified result

$$\tilde{G}^{[0]}(E)\eta - \eta\tilde{G}^{[0]}(E)^\dagger = -2\pi\mathrm{i}\delta\big(\eta E - \eta H^{[0]}\big) = -2\pi\mathrm{i}\delta\big(EI - H^{[0]}\big). \tag{52}$$

A similar proof can be constructed to show that $S^\dagger S = I^R$.

We remark finally that, in a rigorous analysis of a physical process in a finite time interval (e.g., in a quantum measurement theory), it will be necessary to include the closed-channel states due to the incompleteness of the open-channel states in the Hilbert space.

4. Direct sum of two quantized real scalar fields

In this section, we shall advance a dynamics for a quantum field theory, the state space of which comprises the direct sum of the state spaces for two quantized real (i.e. Hermitian) Klein–Gordon fields. We shall show that the dynamics is relativistically invariant, and work out a simple example of collision dynamics using first-order perturbation theory.

We use the time and space coordinates $x^\mu = (x^0, x^1, x^2, x^3)$, the metric tensor $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ and the Hilbert space notation of section 2, and take both $\mathcal{H}^F$ and $\mathcal{H}^B$ to be copies of Fock space (see [43], ch 7a) for an electrically neutral spin zero particle of mass m. As before, Y can take either value F or B. Let the zero state be $\Upsilon(Y, z) \in \mathcal{H}^Y$, the vacuum state (with Hilbert space norm +1) be $\Upsilon(Y, 0) \in \mathcal{H}^Y$, and let $a^Y_{\boldsymbol{p}}$ and $a^{Y\dagger}_{\boldsymbol{p}}$ be the operators that destroy and, respectively, create a particle of 3-momentum $\boldsymbol{p}$. We normalize these operators such that their commutators are

$$\left[a_{\boldsymbol{p}}^{Y}, a_{\boldsymbol{p}'}^{Y}\right] = 0 \tag{53a}$$

$$\left[a_{\boldsymbol{p}}^{Y\dagger}, a_{\boldsymbol{p}'}^{Y\dagger}\right] = 0 \tag{53b}$$

$$\left[a_{\boldsymbol{p}}^{Y}, a_{\boldsymbol{p}'}^{Y\dagger}\right] = \delta^3(\boldsymbol{p} - \boldsymbol{p}')I^Y. \tag{53c}$$

Now let $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N\}$ be a finite set of distinct 3-momenta; then we have the following state in $\mathcal{H}^Y$ that corresponds to one particle with 3-momentum $\boldsymbol{p}_1, \ldots,$ and one particle with 3-momentum $\boldsymbol{p}_N$:

$$\Upsilon(Y, N; \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N) = (N!)^{-1/2} a_{\boldsymbol{p}_1}^{Y\dagger} \cdots a_{\boldsymbol{p}_N}^{Y\dagger} \Upsilon(Y, 0). \tag{54}$$

The normalization guarantees that I_N^Y,

$$I_N^Y = \int \cdots \int_{\mathbb{R}^3} \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_N \Upsilon(Y, N; \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N)\Upsilon(Y, N; \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N)^\dagger \tag{55}$$

is a projection operator from $\mathcal{H}^Y$ to the subspace of N-particle states in $\mathcal{H}^Y$.

Let U^{FB} be the simple linear mapping from $\mathcal{H}^B$ to $\mathcal{H}^F$, in that

$$U^{FB}\Upsilon(B, z) = \Upsilon(F, z) \tag{56a}$$

$$U^{FB}\Upsilon(B, 0) = \Upsilon(F, 0) \tag{56b}$$

$$U^{FB}\Upsilon(B, 1; \boldsymbol{p}_1) = \Upsilon(F, 1; \boldsymbol{p}_1) \tag{56c}$$

$$\vdots \ldots .$$

The operator U^{FB} obviously has a two-sided inverse U^{BF} that coincides with its adjoint, i.e.

$$U^{BF} = (U^{FB})^{-1} = (U^{FB})^\dagger \tag{57a}$$

$$U^{FB}U^{BF} = I^F \tag{57b}$$

$$U^{BF}U^{FB} = I^B. \tag{57c}$$

We reconstruct the subfields in terms of the destruction and creation operators in the manner of Peskin and Schroeder [44], p 21, with $\omega_{\boldsymbol{p}} = \sqrt{k^2 + m^2} > 0$:

$$\phi^Y(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{[2\omega_{\boldsymbol{p}}(2\pi)^3]^{1/2}} \left[a_{\boldsymbol{p}}^Y \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) + a_{\boldsymbol{p}}^{Y\dagger} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x})\right] \tag{58a}$$

$$\pi^Y(\boldsymbol{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3 p \frac{(-\mathrm{i})(\omega_{\boldsymbol{p}})^{1/2}}{[2(2\pi)^3]^{1/2}} \left[a_{\boldsymbol{p}}^Y \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) - a_{\boldsymbol{p}}^{Y\dagger} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x})\right]. \tag{58b}$$

The latter entities have the commutators

$$[\phi^Y(\boldsymbol{x}), \phi^Y(\boldsymbol{y})] = 0 \tag{59a}$$

$$[\pi^Y(\boldsymbol{x}), \pi^Y(\boldsymbol{y})] = 0 \tag{59b}$$

$$[\phi^Y(\boldsymbol{x}), \pi^Y(\boldsymbol{y})] = \mathrm{i}\delta^3(\boldsymbol{x} - \boldsymbol{y})I^Y. \tag{59c}$$

The field operators satisfy

$$\phi^B(\boldsymbol{x}) = U^{BF}\phi^F(\boldsymbol{x})U^{FB} \tag{60a}$$

$$\pi^B(\boldsymbol{x}) = U^{BF}\pi^F(\boldsymbol{x})U^{FB}. \tag{60b}$$

The physical dimensions of the fields $\phi^Y(\boldsymbol{x})$ and $\pi^Y(\boldsymbol{x})$ are (length)$^{-1}$ and (length)$^{-2}$, respectively, modulo powers of $\hbar$ and c.

We shall now formulate a particular case of dynamics and show how to verify that the theory is relativistically invariant. Following the pattern in [44], equations (2.8), (2.18), (2.19) and (4.12), we postulate *ad hoc* the following operator for the energy density $T^{00}(\boldsymbol{x})$:

$$T^{00}(\boldsymbol{x}) = T^{[0]00}(\boldsymbol{x}) + T^{[1]00}(\boldsymbol{x}) \tag{61}$$

where

$$T^{[0]00}(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{2}(\pi^F(\boldsymbol{x})^2 + \nabla_{\boldsymbol{x}}\phi^F \cdot \nabla_{\boldsymbol{x}}\phi^F + m^2\phi^F(\boldsymbol{x})^2) & 0 \\ 0 & \frac{1}{2}(\pi^B(\boldsymbol{x})^2 + \nabla_{\boldsymbol{x}}\phi^B \cdot \nabla_{\boldsymbol{x}}\phi^B + m^2\phi^B(\boldsymbol{x})^2) \end{bmatrix} \tag{62a}$$

$$T^{[1]00}(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{4}\zeta^F\phi^F(\boldsymbol{x})^4 & -\frac{1}{4}\xi\phi^F(\boldsymbol{x})^2 U^{FB}\phi^B(\boldsymbol{x})^2 \\ \frac{1}{4}\xi\phi^B(\boldsymbol{x})^2 U^{BF}\phi^F(\boldsymbol{x})^2 & \frac{1}{4}\zeta^B\phi^B(\boldsymbol{x})^4 \end{bmatrix}. \tag{62b}$$

The dimensionless coupling constants $\zeta^F \geqslant 0$, $\zeta^B \geqslant 0$ and (following, if needed, a separate phase transformation in $\mathcal{H}^F$ and $\mathcal{H}^B$) ξ are all real. The Hamiltonian is defined as follows:

$$H = H^{[0]} + H^{[1]} \tag{63a}$$

$$H^{[0]} = \int_{\mathbb{R}^3} T^{[0]00}(\boldsymbol{x})\,\mathrm{d}^3x \tag{63b}$$

$$H^{[1]} = \int_{\mathbb{R}^3} T^{[1]00}(\boldsymbol{x})\,\mathrm{d}^3x. \tag{63c}$$

The momentum–density operator $T^{j0}(\boldsymbol{x})$ and momentum operator Π^j have forms that do not involve the interaction coupling parameters ζ^F, ζ^B or ξ:

$$T^{j0}(\boldsymbol{x}) = \begin{bmatrix} -\frac{1}{2}\left(\pi^F(\boldsymbol{x})\frac{\partial\phi^F}{\partial x^j} + \frac{\partial\phi^F}{\partial x^j}\pi^F(\boldsymbol{x})\right) & 0 \\ 0 & -\frac{1}{2}\left(\pi^B(\boldsymbol{x})\frac{\partial\phi^B}{\partial x^j} + \frac{\partial\phi^B}{\partial x^j}\pi^B(\boldsymbol{x})\right) \end{bmatrix} \tag{64}$$

$$\Pi^j = \int_{\mathbb{R}^3} T^{j0}(\boldsymbol{x})\,\mathrm{d}^3x. \tag{65}$$

The energy-flow operator $T^{0j}(\boldsymbol{x})$ is taken to be the same operator as $T^{j0}(\boldsymbol{x})$. The stress-tensor operator $T^{jk}(\boldsymbol{x})$ is chosen as follows:

$$T^{jk}(\boldsymbol{x}) = T^{[0]jk}(\boldsymbol{x}) + T^{[1]jk}(\boldsymbol{x}) \tag{66}$$

where

$$T^{[0]jk}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial\phi^F}{\partial x^j}\frac{\partial\phi^F}{\partial x^k} + \frac{1}{2}\delta^{jk}(\pi^F(\boldsymbol{x})^2 - \nabla_{\boldsymbol{x}}\phi^F \cdot \nabla_{\boldsymbol{x}}\phi^F - m^2\phi^F(\boldsymbol{x})^2) & 0 \\ 0 & \frac{\partial\phi^B}{\partial x^j}\frac{\partial\phi^B}{\partial x^k} + \frac{1}{2}\delta^{jk}(\pi^B(\boldsymbol{x})^2 - \nabla_{\boldsymbol{x}}\phi^B \cdot \nabla_{\boldsymbol{x}}\phi^B - m^2\phi^B(\boldsymbol{x})^2) \end{bmatrix} \tag{67a}$$

$$T^{[1]jk}(\boldsymbol{x}) = \begin{bmatrix} -\frac{1}{4}\delta^{jk}\zeta^F\phi^F(\boldsymbol{x})^4 & \frac{1}{4}\xi\delta^{jk}\phi^F(\boldsymbol{x})^2U^{FB}\phi^B(\boldsymbol{x})^2 \\ -\frac{1}{4}\delta^{jk}\xi\phi^B(\boldsymbol{x})^2U^{BF}\phi^F(\boldsymbol{x})^2 & -\frac{1}{4}\delta^{jk}\zeta^B\phi^B(\boldsymbol{x})^4 \end{bmatrix}. \tag{67b}$$

We can now define several other operators on the space of time-dependent states, to assemble a set of generators for the Poincaré group and Schrödinger equation:

$$\Pi^0 = \frac{1}{\mathrm{i}}\frac{\partial}{\partial x^0}I \tag{68a}$$

$$\Omega = \Pi^0 + H \tag{68b}$$

$$L^j = \int_{\mathbb{R}^3} \epsilon^{jkl}x^kT^{l0}(\boldsymbol{x})\,\mathrm{d}^3x \tag{68c}$$

$$B^j = x^0\Pi^j - \int_{\mathbb{R}^3} x^jT^{00}(\boldsymbol{x})\,\mathrm{d}^3x. \tag{68d}$$

The rotation generators L^j and Lorentz 'boost' generators B^j have been defined as in [45], equations (11.57) and (15.19). We call Ω the Schrödinger operator, as the Schrödinger equation for the time-dependent state $\Phi(x^0) \in \mathcal{H}^F \oplus \mathcal{H}^B$ is

$$\Omega\Phi(x^0) = 0. \tag{69}$$

The real linear span of the set $\mathcal{P}$ of ten operators

$$\mathcal{P} = \{-H, \{\Pi^j, L^j, B^j, \text{ for } j = 1, 2, 3\}\} \tag{70}$$

comprises a Lie algebra that is isomorphic to that of the Poincaré group, as is verified by computing the following commutators (we omit the calculational details):

$$[\Pi^j, H] = 0 \tag{71a}$$

$$[L^j, H] = 0 \tag{71b}$$

$$[B^j, H] = -\mathrm{i}\Pi^j \tag{71c}$$

$$[\Pi^j, \Pi^k] = 0 \tag{71d}$$

$$[L^j, \Pi^k] = \mathrm{i}\epsilon^{jkl}\Pi^l \tag{71e}$$

$$[B^j, \Pi^k] = -\mathrm{i}\delta^{jk}H \tag{71f}$$

$$[L^j, L^k] = \mathrm{i}\epsilon^{jkl}L^l \tag{71g}$$

$$[L^j, B^k] = \mathrm{i}\epsilon^{jkl}B^l \tag{71h}$$

$$[B^j, B^k] = -\mathrm{i}\epsilon^{jkl}L^l. \tag{71i}$$

We also have the following commutators with Π^0:

$$[\Pi^0, H] = 0 \tag{72a}$$

$$[\Pi^0, \Pi^j] = 0 \tag{72b}$$

$$[\Pi^0, L^j] = 0 \tag{72c}$$

$$[\Pi^0, B^j] = -\mathrm{i}\Pi^j. \tag{72d}$$

It proves to be the case that Ω commutes with all ten basis elements and hence all elements of the Poincaré group's Lie algebra,

$$[X, \Omega] = 0 \qquad \text{for all} \quad X \in \mathcal{P} \tag{73}$$

and all elements of the component of the identity of the Poincaré group are obtained by exponentiating $(-\mathrm{i})$ times some element of the Lie algebra. Hence, the dynamics entailed by the Schrödinger equation is invariant under the component of the identity of the Poincaré group, in the sense that the application of any element of the group to a solution of the equation of motion yields a transformed state that is also a solution to the same equation of motion, i.e. equation (69). We shall not consider the discrete transformations of space and time herein, except to note that, since in general $\zeta^F \neq \zeta^B$, time reversal—in the strict sense of a simple interchange of FMT and BMT—need not be a symmetry of the above dynamics; the latter assertion should be distinguished from symmetry under conventional time reversal, however, which is more accurately termed 'reversal of the direction of (spatial) motion'—see [46] p 325. In the present context, a distinction between reversal of time and reversal of motion can be meaningful, and thereby determines an absolute direction of time—see [12], p 3, footnote 1.

A similarity transformation by the operator W of an operator X is defined as WXW^{-1}. We define the pseudounitary operator W as

$$W = \begin{bmatrix} I^F \cosh\theta & U^{FB} \sinh\theta \\ U^{BF} \sinh\theta & I^B \cosh\theta \end{bmatrix} \tag{74}$$

where θ is a real constant. A similarity transformation by W leaves the rhs of equation (62a) unchanged, and transforms the rhs of equation (62b) into another operator of the same form with different coupling constants. If the discriminant

$$D = (\zeta^F - \zeta^B)^2/4 - \xi^2 \tag{75}$$

is positive, and we choose

$$\theta = -\tfrac{1}{2}\,\mathrm{arctanh}[2\xi/(\zeta^F - \zeta^B)] \tag{76}$$

the resultant operator is block diagonal, i.e. there is no coupling between FMT and BMT (as redefined). Hence we need $\xi \neq 0$ and D nonpositive to guarantee a nontrivial dynamics. A simplification also occurs if D is negative and

$$\theta = -\tfrac{1}{2}\,\mathrm{arctanh}[(\zeta^F - \zeta^B)/2\xi]. \tag{77}$$

The modified coupling constants then have equal diagonal coefficients.

A further remark: in the above kinematics, there is a family of vacuum states given by $\alpha\Upsilon(F,0) \oplus \beta\Upsilon(B,0)$, with α and β being complex constants (at least one of which is nonzero) modulo equivalence by an overall nonzero complex multiplier. Hence the geometry of the space of rays of vacuum states is $\mathbb{C}P^1$, which is homeomorphic to the Riemann sphere, i.e. S^2, see [47], p 22. This fact will be used in section 5.

To complete this section, we shall apply first-order perturbation theory to the above formalism to estimate the cross section for an input state of two particles, both in FMT or both in BMT, to scatter into an output state of two particles, where the two-particle output may be either jointly in FMT or jointly in BMT. First-order perturbation theory consists in substituting $H^{[1]}$ for $T(E)$ in equation (98). After dropping several divergent self-energy terms, we find the result as given in [44], p 112. We work in the CM frame so that the input particles have momenta $+\hat{\boldsymbol{p}}_{\mathrm{in}}|\boldsymbol{p}|$ and $-\hat{\boldsymbol{p}}_{\mathrm{in}}|\boldsymbol{p}|$, the output particles have momenta $+\hat{\boldsymbol{p}}_{\mathrm{out}}|\boldsymbol{p}|$ and $-\hat{\boldsymbol{p}}_{\mathrm{out}}|\boldsymbol{p}|$, and so that the total energy E_{CM} and relative speed v_{rel} (as defined in [48], equation (3.4.18)) are

$$E_{\mathrm{CM}} = 2\omega_{\boldsymbol{p}} \tag{78a}$$

$$v_{\mathrm{rel}} = 2|\boldsymbol{p}|/\omega_{\boldsymbol{p}}. \tag{78b}$$

Then the total cross sections in ordinary units are

$$(\sigma_{\text{total}})_{\text{FMT}\leftarrow\text{FMT}} = \frac{9(\zeta^F \hbar c)^2}{8\pi E_{\text{CM}}^2} \tag{79a}$$

$$(\sigma_{\text{total}})_{\text{BMT}\leftarrow\text{BMT}} = \frac{9(\zeta^B \hbar c)^2}{8\pi E_{\text{CM}}^2} \tag{79b}$$

$$(\sigma_{\text{total}})_{\text{BMT}\leftarrow\text{FMT}} = \frac{9(\xi \hbar c)^2}{8\pi E_{\text{CM}}^2} = (\sigma_{\text{total}})_{\text{FMT}\leftarrow\text{BMT}}. \tag{79c}$$

By way of a numerical estimate, suppose that $\hbar c/E_{\text{CM}}$ is half the π meson Compton wavelength, that is about 10^{-15} m, and that ξ is about 10^{-10}; ζ^F and ζ^B can be large, so long as $D \leqslant 0$ is satisfied. The cross sections of equation (79c) are then about 10^{-50} m^2. These processes are sufficiently unlikely that they are practically unobservable on a microscopic scale, similar to most gravitation-induced phenomena.

Note that a collision in which either FMT$\leftarrow$ FMT or BMT$\leftarrow$ FMT can take place will entail, on average, an apparent violation of conservation laws. At a time earlier than the collision, the quantum state appears to be a superposition, or a kind of mixture, of FMT states and BMT states with equal total energies and momenta. The small BMT component of the state is part of the output, so that we do not, and by our rules cannot, control this part of the temporally initial state. This BMT component of the temporally earlier quantum state looks to our imagination like a probability-amplitude wave converging—as our time increases—on the collision event in spacetime. This wave interacts very weakly with the constituents of the local environment (the laboratory, the earth, etc, all of which are in FMT), even if this wave describes particles as π^0 mesons that, were they to appear in an FMT state, would interact strongly with the same environment. Hence to a first approximation we need not question the fate of this output BMT wave in the past; it will be effectively undetectible to us. (A collision and a detection amount to a second-order process.) Nevertheless, the BMT π^0 would presumably each decay into two BMT gamma rays at a time earlier than the collision, entities which are not treated in the present theory, but which would also interact weakly with the FMT environment. What would be observable after the collision in an FMT laboratory is that there is a small probability that the input particles, including all their energy and momentum, disappear. There would thus be an apparent nonconservation of energy and momentum, as our instruments can conveniently detect only the FMT part of the energy/momentum flow in spacetime. The observed stability of matter could be due to either (1) the smallness of the FMT/BMT coupling or (2) the circumstance that in a hypothetical theory that describes the physical world, fermion (lepton, baryon) quantum numbers associated with FMT and with BMT are separately conserved.

To an extent, then, this theory gives a realization to the popular picture of a time machine for travelling into the past, albeit only on the level of elementary particle physics. The process that a macroscopic entity scatters coherently from an FMT state into a BMT state would be improbable in the extreme.

5. Further discussion and an application

The physical picture that we have adopted amounts to saying that the world can be described by a kinematics that looks like the direct sum of the kinematics of two conventional quantum field theories. We propose the following visualization: the universe consists of a connected spacetime manifold, within which the ingredients of matter can be, besides in the conventional range of FMT physical states, in BMT states; the dynamical coupling, that is the rate of quantum jumping, of matter between these two sets of states is small, but nonzero. What is of physical interest in the context of theory is establishing criteria for determining if transitions between the hypothetical set of BMT states and states in the known FMT world occur at some very low level. Apart from the computation of scattering cross sections in section 4 and remarks on vacuum states later in this section, we shall not deal with this problem herein.

The formalism proposed in section 4 presumes that particles in forward or backward motion have the same bare mass m; the theory satisfies the criterion of relativistic invariance. If we instead introduce distinct bare masses m_F and m_B in equation (62a), relativistic invariance fails. A naive consideration of the possible theoretical structures does not seem to exclude the possibility that the spectrum of masses, spins, electric charges, etc, of elementary particles could be widely different in the FMT and BMT subspaces. But in another circumstance, Weinberg [48], p 145, made the observation that the commutativity restriction for the energy density operator at spacetime points separated by a nonzero space-like interval is the '*. . . condition that makes the combination of Lorentz invariance and quantum mechanics so restrictive*' (italics in the original). There is not yet a counterpart to this condition in the theory described here, as we have avoided the introduction of a Heisenberg picture for field (or any) operators, due in part to the fact that Hamiltonians can have complex eigenvalues, and in part to nonlocal definitions of input and output. The point we want to make is that relativistic invariance may place severe restrictions on the possible mappings from the state space and dynamics of one quantum field theory to those of another, and thereby constrain the differences between possible field physics, and spectrum of particle masses, spins, charges, etc, associated with the respective FMT and BMT sub-worlds. This problem remains to be investigated.

A proposal concerning the existence of matter that has an internally reversed time sense was made by Stannard [49]. The argument there was made in the context of the then-recently-discovered CP-noninvariance of K^0-meson decays, and distinguishes the proposed new kind of matter (called 'Faustian') from conventional antimatter, which was described as ordinary matter moving backward in time. There is a resemblance between the physics of Stannard's Faustian matter and that of matter in BMT proposed herein. However, the paper did not contain a mathematical formulation of the equations of motion of such a generalized system. It may be said that the theory proposed herein is a possible formulation of Stannard's hypotheses, accompanied by the specifications (i) that the state space is the direct sum, rather than a direct product, of the state spaces of matter in FMT and in BMT, and (ii) that the quantum state of the complete system is characterized by joint forward and backward evolution or motion in time from a suitable input combination of initial and final conditions.

Feynman [50] made an attempt to introduce negative probabilities into physics that is distinct from the work cited in [29] on indefinite metrics. We emphasize that in the theory presented here, probabilities are non-negative and S-matrices are unitary as opposed to pseudounitary. The metric of indefinite sign is interpreted as giving rise to a net current of something across a complete space-like surface, where the current is associated with the probability in a way that involves both input and output states. Analogous to spin, the quantity that gives rise to the current is not further described, and these flows are nonclassical: the 'velocity' of flow in spacetime can in effect have only the values +1 and -1, that is the eigenvalues of the metric operator η, corresponding to FMT and BMT, respectively. 'Current', 'flow' and 'transport' in time are taken as physically suggestive words, but we do not, and assert that we need not, specify in the sense of classical mechanics either what it is that is flowing or the existence of any extra parameter with respect to which the rate of flow is defined.

The association of the expectation value of a quantity with the net transport of that quantity is taken as a physical axiom, which has no deeper explanation in the present context.

We have introduced a theoretical construct in which an event, taken as a cause, can have effects either earlier or later than the cause, or both. Concordantly, we adopt what is called the 'block universe' viewpoint by Price ([14], p 12, and the following) and by Nahin ([51], p 150, and the following), of the dynamically prescribed configuration of a system taken as a whole for all space and for all times in a chosen interval. An entity that can control the complete input to, and observe the output from, such a system must in some sense stand outside time and space as we know them, that is must have what is called an 'atemporal Archimedean standpoint' by Price ([14], p 114). This 'outside' standpoint is analogous to that in which an ordinary observer in spacetime can manipulate the input for solutions of the steady-state, time-independent Schrödinger equation. The phenomenon of closed causal chains, in the sense of Reichenbach ([8], p 36) or Nahin ([51], p 196) could arise in this hypothetical universe. Self-consistency of this process apparently requires a kind of determinism, or a limitation on free will, that is in contradiction to our present understanding. The latter problem also arises in the hypothetical case of topologically connected spacetimes with closed time-like world lines—see [15], p 254 or [51], pp 80–3.

A conventional quantum field theory has a unique vacuum state, a circumstance that permits simplifications, e.g. positioning the energy axis so that the vacuum energy is zero. In the field theory of section 4, there are two vacuum states. (We remark that the physical vacuum is also nonunique in gauge theories—see e.g. [52], ch 10—but this results from assuming basic tachyon, or imaginary mass, fields with certain higher-than-second-order potential energy terms in the classical field Lagrangian, such that the unique mathematical vacuum is a local maximum in the field potential energy, and the minimum energy states form points of a degenerate manifold of field states disjoint from this primitive vacuum state; in the present case, we assume that the bare masses are positive, and that the higher-order interaction energy terms give rise to physical vacuum states having complex energy eigenvalues, i.e. are closed channels.) In order to gain physical insight concerning this possibility, we devote the remainder of this section and of the paper to a nonperturbative calculation on vacuum states and energies. With minor modifications, the mathematics that follows could accommodate the vacuum state matrix of any suitable Hamiltonian; to keep to a specific and simple model, we use the Hamiltonian of equation (63). We establish a two-channel problem consisting of the vacuum states

$$\Psi^F = \begin{bmatrix} \Upsilon(F,0) \\ 0 \end{bmatrix} \qquad \Psi^B = \begin{bmatrix} 0 \\ \Upsilon(B,0) \end{bmatrix}. \tag{80}$$

In the time interval $[0, \tau]$, let the normalized input state be (cf equation (21))

$$\Phi_{\text{in}}(0,\tau) = \begin{bmatrix} \Upsilon(F,0)\cos\theta \\ \Upsilon(B,0)\exp(\mathrm{i}\psi)\sin\theta \end{bmatrix} \tag{81}$$

where θ and ψ are polar and azimuthal coordinates, respectively, on S^2. The output state also comprises the direct sum of vacuum states taken at two different times (cf equation (22)),

$$\Phi_{\text{out}}(\tau,0) = \begin{bmatrix} \Upsilon(F,0)\beta^F \\ \Upsilon(B,0)\beta^B \end{bmatrix} \tag{82}$$

where β^F and β^B are complex coefficients that comprise the output data, which we know beforehand must satisfy the normalization condition

$$|\beta^F|^2 + |\beta^B|^2 = 1. \tag{83}$$

We assume a time-dependent state vector $\Psi(t)$ of the form

$$\Psi(t) = \Psi^F \Phi_F(t) + \Psi^B \Phi_B(t) \tag{84}$$

and establish a coupled, first-order differential equation for the time evolution of the coefficient functions $\Phi_Y(t)$, $Y = F, B$. The equations of motion are

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_Y(t) = \alpha_Y \sum_{Y'=F,B} \left(\Psi^Y; H\Psi^{Y'}\right) \Phi_{Y'}(t). \tag{85}$$

The matrix of the Hamiltonian proves to be

$$\left(\Psi^Y; H\Psi^{Y'}\right) = E^{[0]}\alpha_Y \delta^{YY'} + E^{[1]}\left(\alpha_Y \zeta^Y \delta^{YY'} - \xi\delta^{YF}\delta^{BY'} - \xi\delta^{YB}\delta^{FY'}\right). \tag{86}$$

In equation (86), $E^{[0]}$ and $E^{[1]}$ are the conventional (FMT only) vacuum expectation values of the zero-order Hamiltonian and $\frac{1}{4}\int \phi(x)^4\,\mathrm{d}^3x$, respectively; to be sure, both of these quantities are plus infinity in the present theory, but we shall pretend otherwise and see what happens. The eigenvalues of the Hamiltonian matrix are

$$E^Y = E^{[0]} + E^{[1]}\left[\tfrac{1}{2}(\zeta^F + \zeta^B) - \mathrm{i}\alpha_Y\sqrt{-D}\right] \tag{87}$$

where α_Y is defined in equation (36), and we have presumed that the D of equation (75) is negative. We therefore have a coupled-channel problem that is akin to an ordinary single-channel bound state problem in the context of a second-order, time-independent Schrödinger equation; however, there is no energy-like parameter that can be varied here, nor is there a segment of the time axis in which a shift between a rising and a falling exponential can take place. Hence a bound state in the time dimension does not occur in this case.

Continuing the argument, we define

$$\bar{E} = E^{[0]} + E^{[1]}(\zeta^F + \zeta^B)/2 \tag{88a}$$

$$\kappa = (\zeta^F - \zeta^B)/2 \tag{88b}$$

$$\mu = \sqrt{\xi^2 - \kappa^2} = \sqrt{-D} > 0 \tag{88c}$$

$$\cos\sigma = \kappa/\xi \tag{88d}$$

$$\sin\sigma = \mu/\xi. \tag{88e}$$

A set of eigensolutions to the Schrödinger equation (85) is then, for $Y = F, B$,

$$\Phi^{(Y)}(t) = \begin{bmatrix}\Phi_F^{(Y)}(t)\\ \Phi_B^{(Y)}(t)\end{bmatrix} = \exp(-\mathrm{i}\bar{E}t - \alpha_Y\mu E^{[1]}t)\begin{bmatrix}\mathrm{i}\xi\\ \mathrm{i}\kappa - \alpha_Y\mu\end{bmatrix}. \tag{89}$$

The reason for the superscript is that the solution $\Phi^{(F)}(t)$ ($\Phi^{(B)}(t)$) decreases exponentially as $t \to +\infty$ ($t \to -\infty$). The matrices of η and ηH in the latter basis are time independent, and have the values

$$\Phi^{(Y)}(t)^\dagger\eta\Phi^{(Y')}(t) = \delta^{YF}\delta^{BY'}[2\mu(\mathrm{i}\kappa + \mu)] + \delta^{YB}\delta^{FY'}[-2\mu(\mathrm{i}\kappa - \mu)] \tag{90a}$$

$$\begin{aligned}\Phi^{(Y)}(t)^\dagger\eta H\Phi^{(Y')}(t) = {}& \delta^{YF}\delta^{BY'}[2\mu(\mathrm{i}\kappa + \mu)(\bar{E} + \mathrm{i}\mu E^{[1]})]\\ &+ \delta^{YB}\delta^{FY'}[-2\mu(\mathrm{i}\kappa - \mu)(\bar{E} - \mathrm{i}\mu E^{[1]})].\end{aligned} \tag{90b}$$

A general solution to the Schrödinger equation has the form

$$\Phi(t) = C^{(F)}\Phi^{(F)}(t) + C^{(B)}\Phi^{(B)}(t). \tag{91}$$

We choose the constants $C^{(Y)}$ so that the input boundary conditions (81) are satisfied. We find that

$$C^{(Y)} = [-\alpha_Y \sin\theta \exp(\mathrm{i}\psi + \mathrm{i}\bar{E}\tau) + \alpha_Y \cos\theta \exp(-\mathrm{i}\alpha_Y \sigma + \alpha_Y \mu E^{[1]}\tau)] \times \{\mathrm{i}\xi[\exp(-\mathrm{i}\sigma + \mu E^{[1]}\tau) - \exp(+\mathrm{i}\sigma - \mu E^{[1]}\tau)]\}^{-1}. \quad (92)$$

The expectation values, as defined in equation (26), for the unit operator and the Hamiltonian in state $\Phi(t)$ are given by

$$[I]_{\mathrm{Av}} = 2\sin\sigma[-\sin\sigma + 2\sin\sigma\cos(\sigma + \psi + \bar{E}\tau)\cosh(\mu E^{[1]}\tau) - 2\cos\sigma\sin(\sigma + \psi + \bar{E}\tau)\sinh(\mu E^{[1]}\tau)]\Delta^{-1} \quad (93a)$$

$$[H]_{\mathrm{Av}} = \bar{E}[I]_{\mathrm{Av}} + E^{[1]}A\Delta^{-1} \quad (93b)$$

$$\Delta = \cosh(2\mu E^{[1]}\tau) - \cos(2\sigma) \quad (93c)$$

$$A = 2\xi\sin^2\sigma[\cos\sigma - 2\cos\sigma\cos(\sigma + \psi + \bar{E}\tau)\cosh(\mu E^{[1]}\tau) - 2\sin\sigma\sin(\sigma + \psi + \bar{E}\tau)\sinh(\mu E^{1]}\tau)]. \quad (93d)$$

When $\mu E^{[1]}\tau$ is large, we find that

$$[I]_{\mathrm{Av}} \to -4\sin\sigma\sin(\psi + \bar{E}\tau)\exp(-\mu E^{[1]}\tau) + O(\exp(-2\mu E^{[1]}\tau)) \quad (94a)$$

$$[H]_{\mathrm{Av}} \to -4\sin\sigma[\bar{E}\sin(\psi + E^{1]}\tau) + E^{[1]}\xi\sin\sigma\cos(\psi + \bar{E}\tau)]\exp(-\mu E^{[1]}\tau) + O(\exp(-2\mu E^{[1]}\tau)). \quad (94b)$$

Hence, if the 'experiment' is performed over a time interval τ that is sufficiently long, the vacuum expectation values across a time = constant surface of the probability and of the energy are both exponentially small uniformly over the closed time interval $[0, \tau]$. In other words, whatever the input vacuum state, the magnitude and phase of the resulting time-dependent vacuum state will, for a sufficiently long time τ, adjust themselves so that, at any given time, almost equal amounts of probability are in FMT and in BMT, and almost equal amounts of energy are in FMT and in BMT.

Suppose, finally, that we compute the expectation values $\bar{T}^{\mu\nu}(x^0, \boldsymbol{x})$ with respect to $\Phi(x^0)$ of the components of the stress–momentum–energy–density operators $T^{\mu\nu}(\boldsymbol{x})$, $\mu, \nu = 0, 1, 2, 3$, as given in equations (61), (65) and (66):

$$\bar{T}^{\mu\nu}(x^0, \boldsymbol{x}) = \Phi(x^0)^\dagger \eta T^{\mu\nu}(\boldsymbol{x})\Phi(x^0). \quad (95)$$

So long as $\Phi(x^0)$ satisfies the Schrödinger equation (69), the position-dependent array $\bar{T}^{\mu\nu}(x^0, \boldsymbol{x})$ can be shown to have zero four-divergence and to have the transformation properties of a second-rank contravariant tensor field under the action of the restricted Poincaré group, in the sense that the application of one of the Lie algebra elements of equation (71) to $\Phi(x^0)$ yields the same effect on $\bar{T}^{\mu\nu}(x^0, \boldsymbol{x})$ as would have the corresponding Lie algebra element acting on such a tensor field. Accordingly, we can take such a $\bar{T}^{\mu\nu}(x^0, \boldsymbol{x})$ to be the source distribution of a linearized, classical gravitational field in a background Minkowski spacetime. If we choose $\Phi(x^0)$ to be the vacuum state of equations (84), (91) and (92), the result of equation (95) is not a tensor field (in particular, with respect to Lorentz boosts), since the vacuum state does not satisfy the complete Schrödinger equation. Nevertheless, we take the (still divergent) vacuum expectation value $\bar{T}^{00}(x^0, \boldsymbol{x})$ to be an estimate for the energy density due to the vacuum. This energy density amounts to the expectation value for total energy, divided by the total volume of space. We construe the result (94*b*) as contributing to an explanation for the cosmological constant problem, as described in [53] or by Carroll online [54]: given that the discriminant D of equation (75) is negative, the net vacuum energy density in spacetime should have a very small magnitude, and the energy density would depart from zero due mainly to the presence of ordinary matter in FMT or BMT, and possibly to small vacuum effects that do not enter into the present simple theory and approximation. Equation (94) also suggests that if $D < 0$ the probability that the system is found to be in the BMT vacuum state, but not necessarily in states involving matter in BMT, is about the same as the probability of finding the FMT vacuum state.

Appendix. Transition rates

We want to obtain an expression that permits us to deal with the energy delta-functions in equation (47) to obtain transition probabilities per unit time and cross sections. Although the formalism permits inputs at the initial and final times to be coherent, and permits the study of outputs with definite phase relationships between the temporally earlier and temporally later parts of the output, we shall not attempt this level of generality here: we assume phase incoherence between the FMT and the BMT parts of the input, and discard all information on interference between the FMT and the BMT parts of the output. In other words, we shall presume a block-diagonal (FF and BB only) density matrix at input, and discard block off-diagonal (FB and BF) parts of the density matrix at output.

Let us begin with equation (45) with $\gamma' \neq \gamma$, with the adiabatic switching factors $\exp[-\epsilon|t - t_1|]$ inserted in the integrands, and the integrals carried out:

$$\left(\Phi^{[0]R,Y'}_{E'\gamma'}(t);\, \Phi^{R,Y}_{E\gamma}(t)\right) = -\mathrm{i}\left(\Psi^{[0]R,Y'}_{E'\gamma'};\, \eta T(E)\Psi^{[0]R,Y}_{E\gamma}\right)\left[\delta^{Y'F}\frac{\mathrm{i}\exp[\mathrm{i}(E'-E)t]}{(E-E'+\mathrm{i}\epsilon)} + \delta^{Y'B}\frac{\mathrm{i}\exp[\mathrm{i}(E'-E)t]}{(E-E'-\mathrm{i}\epsilon)}\right]. \tag{96}$$

We relate the parameter ϵ to the effective on time τ of the interaction as follows: insofar as the interaction affects the FMT output, we have presumed that the FMT part of Green's function $G^{[0]}(t - t_1)$ is switched on as $\exp[\epsilon(t_1 - t)]$ and therefore has squared magnitude $\exp[2\epsilon(t_1 - t)]$. We have

$$\tau = \int_{-\infty}^{t} \exp[2\epsilon(t_1 - t)]\,\mathrm{d}t_1 = 1/(2\epsilon). \tag{97}$$

To obtain a transition probability per unit time, we shall divide the transition probability, summed over a range in energy of output states, by τ. A similar result is obtained for the effect of the modulated Green's function on the BMT output.

We compute the absolute square of either the FMT ($Y' = F$) or the BMT ($Y' = B$) part of the rhs of equation (96). In both cases, the rhs has a factor $1/[(E - E')^2 + \epsilon^2]$. This factor will be construed as tending to a delta-function in energy as ϵ becomes small, in fact close to $(\pi/\epsilon)\delta(E - E')$. Since $\pi/\epsilon = 2\pi\tau$, the transition probability per unit time becomes, when the sum over output energy states is converted to an integral with a density of states,

$$\frac{2\pi}{\hbar}\left|\left(\Psi^{[0]R,Y'}_{E\gamma'};\, \eta T(E)\Psi^{[0]R,Y}_{E\gamma}\right)\right|^2 \rho^{Y'}_{\gamma'}(E) \tag{98}$$

where $\rho^{Y'}_{\gamma'}(E)$ is a density in energy of output states of type $Y' = F$ or $Y' = B$, state index γ', and energy E.

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